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Bures and statistical distance for squeezed thermal states

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Abstract. We compute the Bures distance between two thermal squeezed states and deduce the statistical distance metric. By computing the curvature of this metric we can identify regions of parameter space most sensitive to changes in these parameters and thus lead to optimum detection statistics.

1. Introduction

There has been increasing interest recently in the geometry of quantum state space [1, 2]. Although some general features of the geometry are known, little has been learned concerning the details of the geometry of high-dimensional pure and impure states. The initial discovery of a geometric phase by Berry [4] was interpreted by Simon [3] as the holonomy transformation in parallel transporting the adiabatic eigenstate in parameter space [5]. Since then the concept of the geometric phase has been broadened to cope with non-adiabatic, non-cyclic and non-unitary evolutions [7]. However, although the formal understanding of the geometric phase has progressed, knowledge of the underlying geometry described by this phase has not. This is mainly due to computational difficulties in calculating metric tensors. These metric tensors are functionals of infinite-dimensional density matrix operators. In this paper we calculate the metric and curvature of the parameter space of squeezed thermal quantum states. Using a basic understanding of the quantum metric from a statistical inference viewpoint we identify regions of parameter space which yield large changes in the quantum state making its determination easier in such a parameter regime. In section 2 we review the work done to date in uncovering the geometry of a quantum state and introduce the Bures metric. The main results of this paper are the method of calculation and final results in section 4. We have tried to keep this paper as short as possible while including enough detail for the reader to be able to reproduce their own calculation.

2. Review

In this section we review the basics of the geometry of quantum states and introduce concepts associated with the natural extension of the Fubini–Study metric to impure density matrices and, in particular, the Bures metric.

For pure quantum states the geometry is CP^n and is essentially the geometry of the horizontal section of the fibre bundle over the space of pure states with a fibre group

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$U(1)$. The connection defining this section is 'natural' in that the resulting metric and distance functions are invariant under a global change in phase of the states involved. More precisely, the expectation value of any operator in the state $|\psi\rangle$ is unchanged under the action $|\psi\rangle \to \mathrm{e}^{\mathrm{i}\theta}|\psi\rangle$. We can thus split up the space of pure states into conjugacy classes under the $U(1)$ action and denote the class formed from the state $|\psi\rangle$ as the *ray* at $|\psi\rangle$. We can define a distance between two rays as the smallest transition probability between any two elements in the separate rays, i.e. $D^2_{\mathrm{FS}} = \inf \| |\psi_1\rangle - \mathrm{e}^{\mathrm{i}\theta}|\psi_2\rangle\|^2$. Extremizing over the relative phase θ we obtain the well known Fubini–Study distance for pure states $D^2_{\mathrm{FS}} = 2(1 - |\langle\psi_1|\psi_2\rangle|^2)$. One can show that the geometry is Kähler and the metric, $\mathrm{d}s^2_{\mathrm{FS}}$, is the Hessian of a suitable Kähler potential [6]. This Riemannian metric arises in calculations of Berry's phase and generalizations thereof [7] and in aspects of quantum distinguishability [12]. The geometry of *impure* quantum states has received little attention. A Riemannian metric for classical probability distributions was obtained independently by Wootters [8] and Campbell [9]. A transition amplitude between two impure quantum states was discovered by Bures [10]. This amplitude and the related metric has been studied at length with regard to geometric phase by Uhlmann [11]. The same metric has also been obtained in other work relating to the optimal statistical distinguishability between two quantum states [12].

Although some formal work has been done on the geometry of impure quantum states, few concrete results concerning the details of the metric have been found. This is due to the technical difficulties in computing the Bures (or statistical distance) metric. Before giving the formula we will outline briefly the origins of this metric following Uhlmann. The derivation follows the above argument for the Fubini–Study metric on pure states. Beginning with an impure state ρ one *purifies* this state by enlarging the Hilbert space into a Hilbert–Schmidt space through the 'square root' of ρ, i.e. $\mathcal{H} \to \mathcal{H}^{\mathrm{ext}} \equiv \mathcal{H} \otimes \mathcal{H}^*$ where $\rho \equiv WW^*$, W is pure in $\mathcal{H}^{\mathrm{ext}}$ and where $\mathrm{Tr}\, WW^* < \infty$. The 'square root' W, of ρ is defined up to right multiplication by an arbitrary unitary operator V. We again have a fibre bundle structure with base space $\sqrt{\rho}$ and fibre R_V, where R_V is right multiplication by V. The natural distance in $\mathcal{H}^{\mathrm{ext}}$ is the Hilbert–Schmidt metric $\mathrm{d}^2(W_1, W_2) \equiv \mathrm{Tr}(W_1 - W_2)(W_1 - W_2)^*$. This gives a natural connection on the bundle and one can again define a distance between two fibres to be the smallest Hilbert–Schmidt distance between elements of the fibres. The solution to the extremization of $W(\lambda)$, where λ is an affine parameter, is $\dot{W} = GW$ where $G = G^*$ and $\dot{} \equiv \mathrm{d}/\mathrm{d}\lambda$. The induced metric on the horizontal section is just

$$(\dot{W}, \dot{W})_{\mathrm{HS}} = (GW, GW)_{\mathrm{HS}} = \mathrm{Tr}\, G^2\rho = \tfrac{1}{2}\,\mathrm{Tr}\, G\dot{\rho}\,. \tag{1}$$

The extremized $W(\lambda)$ is parallel transported with respect to the natural connection and gives rise to an evolution for $\rho(\lambda)$ in $\mathcal{H}$ which obeys

$$\dot{\rho} = G\rho + \rho G. \tag{2}$$

The Bures distance which results from the extremization can be written as

$$D^2_{\mathrm{B}}(\rho_1, \rho_2) = 2\left[1 - \mathrm{Tr}\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}\right] \tag{3}$$

while the infinitesimal Riemannian metric resulting from this distance is

$$\mathrm{d}s^2_{\mathrm{B}} \equiv \mathrm{Tr}\, G^2\rho = \tfrac{1}{2}\,\mathrm{Tr}\, G\,\mathrm{d}\rho \tag{4}$$

where

$$\mathrm{d}\rho = G\rho + \rho G. \tag{5}$$

This metric is also known as the statistical distance metric and is symmetric in ρ_1 and ρ_2 [12].

Investigation into the detailed structure of the Bures distance has been hampered by the complicated square-root factors in (3). The distance and metric have been calculated for the spin-$\frac{1}{2}$ system [13] and the spin-1 system [14, 15]. It was found that the geometry of the spin-$\frac{1}{2}$ state space was of constant curvature. However, the geometry of the spin-1 state space possessed a non-constant curvature. It was further proved in [16] that the geometry of state space for spin-n is not of constant curvature and not even locally symmetric for $2n + 1 \geqslant 3$. To directly solve for the Riemannian metric (4) one must solve the matrix Lyapunov equation (5). For quantum systems possessing a finite-dimensional representation the method of annihilating polynomials can be used to solve the Lyapunov equation [17]. This becomes prohibitive for $n > 3$ and results in non-unique expressions with respect to the parametrizations chosen for $\delta\rho$. Other solution methods are available but are again difficult to compute, i.e. recursive solutions.

In the following we will first calculate the Bures distance between two undisplaced thermal squeezed states and from this derive the associated Riemannian metric. Calculating the curvature of this metric we find the space is not of constant curvature and can interpret this curvature as a measure of optimal quantum distinguishability between the states.

3. Bures distance

From the work of Bures and Uhlmann [10, 11] the transition amplitude between two quantum states may be written as

$$D_{\mathrm{B}}^2(\rho_1, \rho_2) = 2\left[1 - \mathrm{Tr}\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}\right]. \tag{6}$$

Due to the complexities of computing the trace, studies of this transition amplitude have concentrated only on finite-dimensional examples with concrete results for dimension 2 [14] and 3 [15]. In the following we will compute the transition amplitude between two thermal squeezed states with density matrices parametrized in the form

$$\rho(\beta, r, \theta) = ZS(r, \theta)T(\beta)S^\dagger(r, \theta) \tag{7}$$

where

$$S(r, \theta) = \exp(\zeta K_+ - \zeta^* K_-)$$
$$T(\beta) = \exp(-\beta K_0)$$
$$\zeta = r\mathrm{e}^{\mathrm{i}\theta}$$

and

$$K_+ = \tfrac{1}{2}a^{\dagger 2} \qquad K_- = \tfrac{1}{2}a^2 \qquad K_0 = \tfrac{1}{2}(a^\dagger a + \tfrac{1}{2})$$
$$[K_0, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_0\,.$$

Here $S(r, \theta)$ is the one-photon squeeze operator, a is the single-mode annihilation operator, Z is chosen so that $\mathrm{Tr}(\rho) = 1$, and $(K_0, K_\pm)$ are the generators of the $SU(1, 1)$ group. Equation (7) thus represents an undisplaced squeezed thermal state. We have written the density matrix in the Schur form $\rho = UTU^\dagger$ where U is unitary and T is diagonal in the eigenbasis of K_0. This decomposition is relatively straightforward in the case of Gaussian ρ [18]. However, we find that the following arguments do not seem to hold if we expand the states considered to include displaced thermal squeezed states. We are thus restricted to density matrices continuously parametrized by three variables (β, r, θ).

In the following this Schur factorization will play a central role. With this factorization we can easily define the square root of a positive operator. A possible alternative method is

to represent the squeeze and thermal operators as 2×2 matrix representations of $SU(1,1)$. However, it is unclear to the author at this time how one can consistently define the square root in this representation without again forming the Schur decomposition of the matrix representation. We now outline how the Schur decomposition yields the square root of a positive operator ρ.

Through an insertion of unity, the Schur factorization of ρ yields a diagonal representation of the state. Choosing orthogonal eigenstates $|\lambda_i\rangle$, such that $T(\beta)|\lambda_i\rangle = \lambda_i(\beta)|\lambda_i\rangle$ and $Z\sum_i \lambda_i = 1$, we can insert the resolution of unity $\sum_i |\lambda_i\rangle\langle\lambda_i| \equiv \mathbb{1}$ into (7) to get

$$\begin{aligned}\rho &= ZUTU^\dagger = ZUT\mathbb{1}U^\dagger \\ &= Z\sum_i UT|\lambda_i\rangle\langle\lambda_i|U^\dagger = Z\sum_i \lambda_i U|\lambda_i\rangle\langle\lambda_i|U^\dagger \\ &= Z\sum_i \lambda_i|u_i\rangle\langle u_i|\end{aligned}$$

where $|u_i\rangle \equiv U|\lambda_i\rangle$, $\langle u_i|u_j\rangle = \delta_{ij}$ and $\mathrm{Tr}(\rho) = 1$. Thus we have diagonalized ρ over a complete orthonormal set of states with corresponding probabilities $P_i = Z\lambda_i$. It is now an easy matter to find $\rho^{1/2}$:

$$\rho^{1/2} = W = \sum_i P_i^{1/2}|u_i\rangle\langle u_i|V \tag{8}$$

where V is an arbitrary unitary operator and $\rho = WW^\dagger$. Essentially, V encodes the ambiguity in taking the square root of an infinite-dimensional operator. To evaluate the trace in (6) we need only compute $\rho_1^{1/2}$. From the alternate definition of the Bures distance

$$D_{\mathrm{B}}^2 = (\sqrt{\rho_1}, \sqrt{\rho_2})_{\mathrm{HS}} = \inf \mathrm{Tr}(W_1 - W_2)(W_1 - W_2)^\dagger \tag{9}$$

where $\rho_1 = W_1W_1^\dagger$, $\rho_2 = W_2W_2^\dagger$, we see that the definition (6) is invariant under the transformation $W_i \to W_i\tilde{V}$ where $\tilde{V}\tilde{V}^\dagger = 1$. By right multiplying W_1 and W_2 by $V_1^\dagger$ in (6) we can shift away the V_1 dependence of $\sqrt{\rho}_1$ to get

$$\rho_1^{1/2} = \sum_i P_i|u_i\rangle\langle u_i|\,. \tag{10}$$

Since we need not calculate $\rho_2^{1/2}$ we can ignore the unitary $V_2V_1^\dagger$ appearing in $\rho_2^{1/2}$.

Let us now summarize the manipulations needed to calculate the complicated trace factor in (6). Given ρ_1 we can now compute $\sqrt{\rho}_1$ taking the positive section for the square root. Using ρ_2 we form the operator $A = \rho_1^{1/2}\rho_2\rho_1^{1/2}$ and, using the Baker–Campbell–Hausdorff identities, we rearrange A into Schur form

$$A = U_A T_A U_A^\dagger\,. \tag{11}$$

In this representation we can easily compute the square root, again taking the positive section. All that remains is to take the trace. The rearrangement of A into Schur form is not trivial. The operators U_A and T_A can be found only in the case of undisplaced squeezed thermal states. For displaced states a Schur resolution was not found through BCH disentangling. It may be that in the case of displaced states the positive section for the square root is not the correct ansatz and the more general form (8) is needed. We will not address this here and will only consider states of the form (7) which do result in Schur decompositions for A.

We now calculate $\mathrm{Tr}\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}$ where

$$\rho_i = Z_i S(r_i, \theta_i)\exp(-\beta_i K_0)S^\dagger(r_i, \theta_i) \tag{12}$$

with normalization $\mathrm{Tr}(\rho) = 1$ or $Z = 2\sinh\beta_i/4$. Writing $T_i = \exp(-\beta_i K_0)$ we must rearrange $\rho_1^{1/2}\rho_2\rho_1^{1/2}$ to have the Schur form

$$\begin{aligned}\rho_1^{1/2}\rho_2\rho_1^{1/2} &= Z_1 Z_2 S_1 T_1^{1/2} S_1^\dagger S_2 T_2 S_2^\dagger S_1 T_1^{1/2\dagger} S_1^\dagger \\ &= Z_1 Z_2 S_1 S_3 T_3 S_3^\dagger S_1^\dagger \equiv A\,. \end{aligned} \tag{13}$$

Taking the positive square root and trace gives

$$\begin{aligned}\mathrm{Tr}\sqrt{A} &= \sqrt{Z_1 Z_2}\,\mathrm{Tr}\,\mathrm{e}^{-\beta_3/2K_0} \\ &= \frac{\sqrt{\sinh\beta_1/4\sinh\beta_2/4}}{\sinh\beta_3/8}\,. \end{aligned} \tag{14}$$

We must now use Baker–Campbell–Hausdorff identities to express β_3 in terms of $(\beta_1, \beta_2, r_1, r_2, \theta_1, \theta_2)$. This first step is to collapse the product $S_1^\dagger S_2$ in (13) into a single squeeze operator. This is accomplished through the identity [20]

$$S^\dagger(r_1,\theta_1)S(r_2,\theta_2) = \mathrm{e}^{-\mathrm{i}\phi}\bar{S}(\bar{r},\bar{\theta}-\phi)\bar{R}(\phi) \tag{15}$$

where $R(\phi) = \mathrm{e}^{\mathrm{i}\phi K_0}$ is the rotation operator while the parameters (r_i, θ_i) are related to $(\bar{r}, \bar{\theta}, \phi)$ through

$$C_{\bar{r}\bar{\theta}}\mathrm{e}^{\mathrm{i}\phi\sigma_3} = C_{r_2\theta_2}C_{r_1\theta_1} \tag{16}$$

where

$$C_{r\theta} = \begin{bmatrix} \cosh r & \mathrm{e}^{2\mathrm{i}\theta}\sinh r \\ \mathrm{e}^{-2\mathrm{i}\theta}\sinh r & \cosh r \end{bmatrix} \tag{17}$$

and σ_3 is the third Pauli matrix. Collapsing this product in (13) gives

$$\rho_1^{1/2}\rho_2\rho_1^{1/2} = Z_1 Z_2 S_1 T_1^{1/2}\bar{S}\bar{R}T_2\bar{R}^\dagger\bar{S}^\dagger T_1^{1/2} S_1^\dagger\,. \tag{18}$$

However, $\bar{R}T_2\bar{R}^\dagger = T_2$ since $\bar{R}$, T_2 commute. The factor $\exp(\mathrm{i}\phi)$ cancels since it is a scalar. We thus have

$$\rho_1^{1/2}\rho_2\rho_1^{1/2} = Z_1 Z_2 S_1 T_1^{1/2}\bar{S}T_2\bar{S}^\dagger T_1^{1/2} S_1^\dagger\,. \tag{19}$$

We must now rearrange the product $T_1^{1/2}\bar{S}T_2\bar{S}^\dagger T_1^{1/2}S_1^\dagger$ into Schur form. To do this we use Baker–Campbell–Hausdorff disentangling. The particular method we use was outlined in [19]. Using the faithful group representation of $SU(1,1)$ where

$$K_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad K_- = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \qquad K_0 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{20}$$

we can express the operator $\bar{S}$ as

$$\bar{S} = \exp(\bar{\zeta}K_+ - \bar{\zeta}^* K_-) = \begin{bmatrix} \cosh\bar{\gamma} & \mathrm{e}^{\mathrm{i}\bar{\theta}}\sinh\bar{\gamma} \\ \mathrm{e}^{-\mathrm{i}\bar{\theta}}\sinh\bar{\gamma} & \cosh\bar{\gamma} \end{bmatrix} \tag{21}$$

where $\bar{\gamma} = \sqrt{\bar{\zeta}\bar{\zeta}^*}$ and $\bar{\theta} = \sqrt{\bar{\zeta}/\bar{\zeta}^*}$. The operator T_i is represented as

$$T_i = \begin{bmatrix} \mathrm{e}^{-\beta_i/2} & 0 \\ 0 & \mathrm{e}^{\beta_i/2} \end{bmatrix}. \tag{22}$$

To re-express the product $T_1\bar{S}T_2\bar{S}^\dagger T_1$ as $ST_3S^\dagger$ we represent each operator as a 2×2 matrix, multiply and compare the resulting entries to obtain

$$\cosh\beta_3/2 = \cosh^2\bar{\gamma}\cosh((\beta_1+\beta_2)/2) - \sinh^2\bar{\gamma}\cosh((\beta_1-\beta_2)/2)\,.$$

Denoting

$$Y = \cosh^2\beta_3/4\cosh^2\bar{\gamma}\cosh^2((\beta 1+\beta_2)/4)) - \sinh^2\bar{\gamma}\cosh^2((\beta_1-\beta_2)/4)$$

and inserting this back into (14) we get

$$\mathrm{Tr}\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}} = \frac{\sqrt{2\sinh\beta_1/4\sinh\beta_2/4}}{\sqrt{\sqrt{Y}-1}}\,. \tag{23}$$

All that remains is to express $\bar{\gamma}$ in terms of $(r_1, r_2, \theta_1, \theta_2)$. From equation (17) we obtain

$$\cosh^2\bar{\gamma} = \cos^2\Delta\theta\cosh^2\Delta r + \sin^2\Delta\theta\cosh^2\Sigma r \tag{24}$$

where $\Delta\theta = \theta_1 - \theta_2$, $\Delta r = r_1 - r_2$ and $\Sigma r = r_1 + r_2$. Defining $\beta_+ = (\beta_1+\beta_2)/4$ and $\beta_- = (\beta_1-\beta_2)/4$ we finally get

$$\begin{aligned} Y &= \cosh^2\beta_3/4 \\ &= \cos^2\Delta\theta[\cosh^2\Delta r\cosh^2\beta_+ - \sinh^2\Delta r\cosh^2\beta_-] \\ &\quad + \sin^2\Delta\theta[\cosh^2\Sigma r\cosh^2\beta_+ - \sinh^2\Sigma r\cosh^2\beta_-] \end{aligned}$$

and

$$\mathrm{Tr}\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}} = \frac{2\sinh\beta_1/4\sinh\beta_2/4}{\sqrt{\sqrt{Y}-1}}\,. \tag{25}$$

For the case $\Delta\theta = \Delta r = 0$, i.e. only a change in temperature, equation (25) gives

$$D_{\mathrm{B}}^2(\rho(\beta_1),\rho(\beta_2)) = 2\left[1 - \frac{\sqrt{\sinh\beta_1/4\sinh\beta_2/4}}{\sinh((\beta_1+\beta_2)/8)}\right]. \tag{26}$$

Equation (26) gives the Bures distance between two thermal states with temperatures proportional to $1/\beta_1$ and $1/\beta_2$. The distance function (26) (or more generally (6)) is a proper distance function on the space of states. However, the resulting form (26) is clearly not a distance function arising from a local metric structure defined on the parameter space. The restriction of the Bures distance to pure states, the Fubini–Study distance, is derivable from a local metric, i.e. $D_{\mathrm{FS}}^2 = 2(1-|\langle\psi_1|\psi_2\rangle|^2) = 2\cos\Delta\theta$ where $\Delta\theta$ is the angle between the two Hilbert space vectors $|\psi_1\rangle$, $|\psi_2\rangle$. From Uhlmann's derivation of the Bures distance as the minimum Fubini–Study distance between the purifications of $\sqrt{\rho_1}$ and $\sqrt{\rho_2}$ in the larger Hilbert–Schmidt space we see that the Bures distance arises from a metric structure in this larger Hilbert–Schmidt space. To derive this metric we can proceed in two ways. We can use standard perturbation analysis to evaluate $\mathrm{d}s_{\mathrm{B}}^2 \equiv D_{\mathrm{B}}^2(\rho, \rho+\delta\rho)$. This was done in [14] with the result

$$\delta D_{\mathrm{B}}^2 = \mathrm{d}s_{\mathrm{B}}^2 = \frac{1}{2}\sum_{i\neq j}\frac{\langle u_i|\delta\rho|u_i\rangle\langle u_j|\delta\rho|u_i\rangle}{P_i+P_j} \tag{27}$$

where $|u_i\rangle$ are the eigenstates of ρ. This metric also appears in [12] and is known there as the statistical distance metric. The Bures or statistical distance metric has been mostly studied in the case of pure states [1], while for impure states only a few finite-dimensional examples have been examined [13–15]. Since we know the orthogonal eigenstates of ρ we could compute (27) explicitly. This is simply done in the case $\Delta r = \Delta\theta = 0$ but becomes quite tedious otherwise. Instead we note that

$$\mathrm{d}s_{\mathrm{B}}^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}D_{\mathrm{B}}(\rho(\beta,r,\theta),\rho(\beta+t\delta\beta,r+t\delta r,\theta+t\delta\theta))\bigg|_{t=0}. \tag{28}$$

Using equations (25), (6) and (28) one can eventually obtain

$$\mathrm{d}s_{\mathrm{B}}^2 = \mathrm{d}s_{\mathrm{SD}}^2 = \tfrac{1}{2}[1+\mathrm{sech}\,\beta/2](\mathrm{d}r^2 + \sinh^2(2r)\,\mathrm{d}\theta^2) + \frac{1}{64\sinh^2\beta/4}\,\mathrm{d}\beta^2 \tag{29}$$

where the subscripts indicate Bures or statistical distance. The metric may be simplified somewhat by defining $\exp(-2u) = \tanh\beta/8$ to give

$$\begin{aligned} \mathrm{d}s_{\mathrm{B}}^2 &= \mathrm{d}s_{\mathrm{SD}}^2 \\ &= \tfrac{1}{2}[1+\tanh^2 u](\mathrm{d}r^2 + \sinh^2(2r)\,\mathrm{d}\theta^2) + \mathrm{d}u^2 \,. \end{aligned} \tag{30}$$

As a check we have directly computed the $\mathrm{d}\beta^2$ contribution from (27) while for pure states ($\beta \to \infty$) the metric reduces to the known form [21]. Equipped with the metric (29), one can compute geometrical quantities such as the scalar curvature

$$R_{\mathrm{SD}} = -\frac{8(\cosh^2\beta/4 + 12\sinh^4\beta/4)}{\cosh^2\beta/2} \,. \tag{31}$$

It is interesting to note that the scalar curvature is independent of the 'unitary' parameters r and θ and only depends on the 'non-unitary' parameter β. This may be understood from the work of Dittmann [15]. Dittmann shows that in finite dimensions the geometry of the parameter space is locally isometric to $S^{n-1} \times U(n)/T^n$. The homogeneous submanifold $U(n)/T^n$, which, in our case, is parametrized by $r(\beta)$ and $\theta(\beta)$, has a constant curvature depending only on β. Thus, in the general case, the curvature should only be a function of the invariants β_i [22] of the state ρ.

4. Distinguishability measure

A physical significance can be attributed to the curvature R_{SD} (equation (31)). From a statistical inference viewpoint the statistical distance can be understood as a measure of how well one can, in principle, determine the parameters describing ρ through N arbitrary generalized measurements. For more on this viewpoint see [12, 23]. The error δX in estimating the parameter X by analysing the data obtained from N copies of $\rho(X)$ is bounded by

$$N\langle(\delta X)^2\rangle\left(\frac{\mathrm{d}s}{\mathrm{d}X}\right)^2 \geqslant 1 \tag{32}$$

where $\mathrm{d}s/\mathrm{d}X$ is the rate of change of statistical distance with respect to the parameter X for a single copy of $\rho(X)$. Thus, if two states are separated by a statistical distance of $\mathrm{d}s$ then one must perform at least $N \geqslant 1/\mathrm{d}s^2$ measurements on identically prepared copies to distinguish between the two. To estimate a parameter separation between two states one can make use of (32).

However, to go beyond distinguishing between two states and to provide a complete estimation of a state given a reference state, one can argue that the accuracy of estimation of the *complete* state should be independent of the particular parametrization used. The most natural geometric quantity which is parametrization (coordinate) independent is the scalar curvature of the statistical distance. Following this argument, we can interpret the curvature R_{SD} as the degree of local distinguishability of the complete state, i.e. if R_{SD} is large then few measurements are needed to estimate the parameters of a neighbouring state while if R_{SD} is small, many measurements will be required to estimate the parameters of a neighbouring state. We plot the behaviour of $R_{\mathrm{SD}}(\beta)$ in figure 1. From the form of R_{SD} (equation (31)), we see that the degree of distinguishability diverges as $\beta \to \infty$. Similarly, from (29) we find that the integrated statistical distance between a pure state and *any* impure state diverges. This feature was also seen in a similar calculation by Braunstein and Milburn [23]. As was explained there, the degree of distinguishability, R_{SD}, diverges at the pure state boundary because only one measurement is needed to differentiate between a pure state with $\beta = \infty$

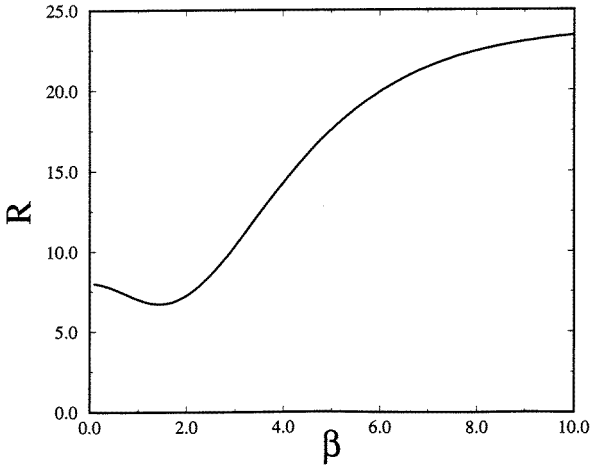

Figure 1. Graph of the curvature of the statistical distance metric R_{SD} versus inverse temperature β in equation (31).

and an impure state with *any* $\beta < \infty$. This principle was the basis of the one-shot clock in [23] where the parameter to be estimated was time. From equation (31) we can also identify local maxima (at $\beta = 0$) and minima (at $\beta = 4\cosh^{-1} 5/\sqrt{22}$). The ultimate accuracy of parameter estimation at various values of β is reflected in the behaviour of the degree of distinguishability R. This measure can serve as a guideline for optimal operating regimes in quantum non-demolition measurements. By targeting the measurements to operate in those regions of high distinguishability one can obtain information about the quantum system in the least number of measurements.

5. Conclusion

In this paper we have examined the geometry of the quantum state. After reviewing the previous work we showed how one can extend the definition of the quantum metric to deal with impure quantum states. The resulting metric is known as the Bures metric. To compute the geometry of a quantum state using this Bures metric necessitates the computation of the square root of a density matrix. Up until now this has only been done for very simple quantum systems. Using a Schur decomposition of ρ we have calculated the Bures distance and associated Riemannian metric between two squeezed thermal states. The Riemannian manifold is not of constant curvature as was suggested by spin-$\frac{1}{2}$ calculations. It was argued that the scalar curvature of the Riemannian manifold can serve as a measure of the ultimate accuracy in determining the parameters defining a quantum state. The method used here can be generalized to more complicated quantum states if a Schur factorization can be found. More generally, computable methods for solving the finite- or infinite-dimensional Lyapunov equations need to be investigated before one can understand the geometric structures of high-dimensional impure quantum states.

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